Higher genus curves and topological recursion

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Plan

- Augmentation varieties for knots in other 3-manifolds
- Consequences of Legendrian SFT
- Construction of Legendrian SFT
- Topological recursion at infinity

Recap

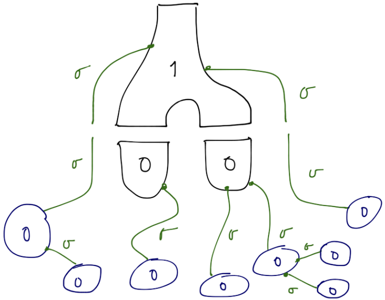
- Recall from yesterday, if X is the resolved conifold, if $L_K \subset X$ the shifted Lagrangian conormal of a knot $K \subset S^3$, and if $W_K(e^x, Q)$ is the GW disk potential then

$$p = \frac{\partial W_K}{\partial x}$$

parameterizes a branch of the augmentation curve V_K of K.

Recap

- The proof used bounding chains to make the usual SFT-augmentation argument work:



and $e^p = e^{\frac{\partial W_K}{\partial x}}$ counts insertions at infinity.

Other manifolds

- From the physical perspective the pair (T^*S^3,resolved conifold) has counterparts for other rational homotopy spheres. Here if M is a geometric quotient of S^3 then the corresponding CY-manifold X_M has one Kähler parameter for each free homotopy class. (We will consider the simplest example $\mathbb{R}P^3$ in concrete calculations.)

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- Unlike in the case S^3 we must consider the input from closed Reeb orbits in this case.
- Here thus the dga $\mathcal{A}(\Lambda_K)$ must be considered as an algebra over the orbit algebra $\mathcal{Q}(M)$ of U^*M.

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- The homology of the algebra $\mathcal{Q}(M)$ is supported in non-positive degrees and the degree 0 part is isomorphic to a commutative algebra generated by the free homotopy classes of loops in M.

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- To see this we use the isomorphism between linearized contact homology and S^1-equivariant symplectic cohomology of the cotangent bundle. We find that over $\mathbb{C}$ we have

$$HC_*^{\text{lin}}(U^*M) = \begin{cases} 0 & \text{if } * < 0, \\ \mathbb{C}\langle \text{hmtpy free loops} \rangle & \text{if } * = 0, \\ 0 & \text{if } * = 1. \end{cases}$$

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- Viewing the linearized homology as the E_2-page for the word length spectral sequence for the full homology we find that there is no differential mapping into the degree 0 part and hence the resulting homology is a commutative algebra.

Other manifolds

- Consider now the counterpart of the arguments above for S^3. We have

$$\mathcal{A}(\Lambda_K) = \mathbb{C}[e^{\pm x}, e^{\pm p}, \gamma_1, \dots, \gamma_m]\langle \mathbf{0}, \mathbf{1}, \dots \rangle$$

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- The isomorphism disks before the transition with positive puncture at γ_j closes up to punctured spheres after the transition, corresponding to "filling the boundary with a disk" and the augmentation polynomial again gives the GW disk potential in terms of the GW-potentials for the γ_j.

Computations

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- We start with the line itself. The result is the augmentation polynomial.

$$A_{\mathbb{R}P^1}(e^x, e^p; Q, \gamma) = e^x - e^{-x} - e^{-x}e^{-p} + Qe^x e^p + \gamma = 0,$$

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- More general links can now be described in a similar local Morse + global disks way as for S^3.

- Consider again the knots in S^3 and $L_K \subset X$. To define Legendrian SFT one should find a framework involving bounding chains so that there is no boundary splittings in more complicated holomorphic curves with boundary. The structure of the theory would then be the following:

Legendrian SFT

- Consider again the knots in S^3 and $L_K \subset X$. To define Legendrian SFT one should find a framework involving bounding chains so that there is no boundary splittings in more complicated holomorphic curves with boundary. The structure of the theory would then be the following:

- There is an SFT-potential $F = F(e^x, Q, g_s)$ that counts configurations of rigid holomorphic curves in X with boundary on L_K, bounding chains, and positive punctures. Note that curves contributing to F must have all positive punctures of degree 0. We have

$$F = F_0 + F_1 + F_2 + \dots,$$

where F_j counts curves with several positive punctures.

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- The boundary of the 1-dimensional moduli spaces then gives the equation

$$e^{-F} H e^{F} = 0, \text{ or simply } H e^{F} = 0.$$

Here we need only consider broken curves with one positive degree 1 chord and the rest degree 0.

Legendrian SFT

- Consider next the counterpart of the substitution $p = \frac{\partial W}{\partial x}$. When counting arbitrary curves we can make any insertions. A coefficient e^p in H then contributes

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- Note that $\Psi_K(x) = e^{F_0}$. Thus using elimination theory in the non-commutative setting $e^p e^x = e^{g_s} e^x e^p$ we should find an operator equation

$$\hat{A}_K(e^x, e^p, Q)\Psi_K(x) = 0,$$

which gives the recursion for the colored HOMFLY.

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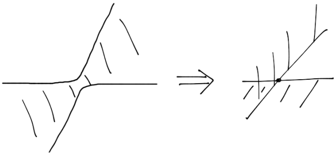
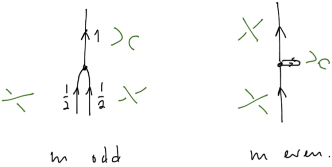
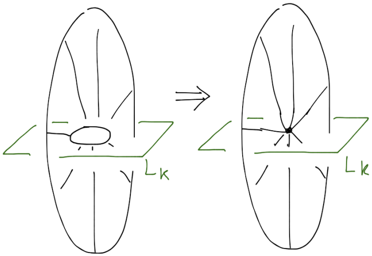
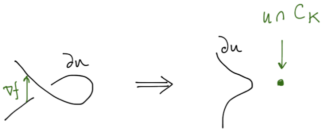
- Let the degree 1 Reeb chords be denoted $b_1, \ldots b_m$ and the degree 0 Reeb chords $a_1, \ldots, a_n$
- Additional data: a Morse function f on L_K which gives obstruction chains. A 4-chain C_K for L_K with $\partial C_K = 2[L_K]$ which looks like $\pm J\nabla f$ near the boundary.

Legendrian SFT



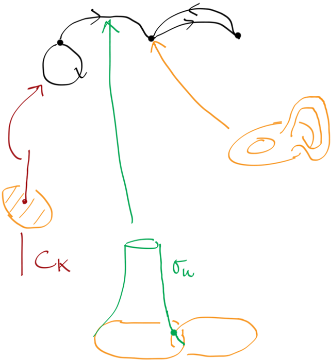






- We define the GW-potential e^F as the generating function of oriented graphs with holomorphic curves at vertices, and intersections with chains at the edges weighted by $\pm\frac{1}{2}$:

$$F = \sum C_{\chi,m,l}\, g_s^{-\chi} e^{mx} a_{i_1} \dots a_{i_r}$$



Legendrian SFT

- Let $H(b_j)$ denote the count of rigid holomorphic curves in the symplectization with a positive puncture at b_j:

$$H(b_j) = \sum C_{\chi,m,n,I,J}\, g_s^{-\chi} e^{mx} e^{np}\, a_{i_1} \dots a_{i_r} g_s^l \frac{\partial}{\partial a_{j_1}} \dots \frac{\partial}{\partial a_{j_l}}.$$

- Then if $p = g_s \frac{\partial}{\partial x}$, $e^{-F} H(b_j) e^F$ counts ends of a 1-dimensional moduli space and in particular:

$$H(b_j) e^F = 0.$$

- We conjecture that the above system give a generator $\widehat{A}_K$ with $\widehat{A}_K e^{F_0} = 0$ for the D-module ideal corresponding to the colored HOMFLY of K (generators $\widehat{A}^j_K$ if K is a link).

- For the unknot there are no (formal) higher genus curves and the operator equation is

$$\widehat{A}_U(e^x, e^p, Q) = (1 - e^x - e^p - Qe^x e^p)\Psi_U = 0.$$

Legendrian SFT and HOMFLY

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- For the Hopf link L Reeb chord generators are as for the trefoil. The relevant parts for the operator H is as follows:

$$H(c_{11}) = (1 - e^{x_1} - e^{p_1} + Qe^{x_1}e^{p_1}) + g_s^2\partial_{a_{12}}\partial_{a_{21}} + \mathcal{O}(a),$$
$$H(c_{22}) = (1 - e^{x_2} - e^{p_2} + Qe^{x_2}e^{p_2}) + Qe^{x_2}e^{p_2}g_s^2\partial_{a_{12}}\partial_{a_{21}} + \mathcal{O}(a),$$
$$H(c_{12}) = (e^{p_2}e^{-p_1} - Qe^{x_2}e^{p_2})g_s\partial_{a_{12}} + g_s^{-1}(e^{-g_s} - 1)(1 - e^{x_2})a_{21} + \mathcal{O}(a^2),$$
$$H(c_{21}) = (Qe^{x_1}e^{p_1} - e^{g_s}e^{p_1}e^{-p_2})g_s\partial_{a_{21}} + g_s^{-1}\left((e^{g_s}(e^{g_s} - 1) - e^{2g_s}(e^{g_s} - 1)e^{x_1})e^{p_1}e^{-p_2} + (e^{g_s} - 1)Qe^{x_1}e^{p_1}g_s^2\partial_{a_{12}}\partial_{a_{21}}\right)a_{12} + \mathcal{O}(a^2).$$

Legendrian SFT and HOMFLY

- After the change of variables,

$$e^{x_1'} = e^{g_s} e^{x_1}, \quad e^{p_1'} = e^{g_s} e^{p_1};$$
$$e^{x_2'} = Q^{-1} e^{-x_2}, \quad e^{p_2'} = e^{-g_s} Q^{-1} e^{-p_2};$$
$$Q' = e^{g_s} Q, \quad g_s' = -g_s,$$

we find the D-module ideal generators

$$\widehat{A}_L^1 = (e^{x_1} - e^{x_2}) + (e^{p_1} - e^{p_2}) - Q(e^{x_1} e^{p_1} - e^{x_2} e^{p_2})$$
$$\widehat{A}_L^2 = (1 - e^{-g_s} e^{x_1} - e^{p_1} + Q e^{x_1} e^{p_1})(e^{x_1} - e^{p_2})$$
$$\widehat{A}_L^3 = (1 - e^{-g_s} e^{x_2} - e^{p_2} + Q e^{x_2} e^{p_2})(e^{x_2} - e^{p_1}),$$

in agreement with HOMFLY.

Legendrian SFT and HOMFLY

- Similarly, for the trefoil we get the D-module ideal generator

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\widehat{A}_T = {} & (Q^2 - Qe^p)e^{g_s}(Q - e^{g_s}e^{2p})Q^2 \\
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& + \left((qe^{2p} - e^p)e^{3g_s}(Q - e^{3g_s}e^{2p})Q^2\right) e^x \\
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- Note that the classical limit $g_s \to 0$ of this is

$$(Q - e^{2p})A_T(e^x, e^p, Q).$$

Topological recursion

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- If the above works then one should be able to construct $\Psi_K(x)$ up to a number of initial conditions from H. We next show how that would work.
- Note first that $W_K(x)$ is given by solving an algebraic equation and hence an analytic function.

Topological recursion

- Using this and the curve counting isomorphism map

$$Ch^{\mathrm{lin}}(\Lambda_K) \oplus C_*(K) \;\to\; \mathrm{Cone}(C_*(\Omega(K,K),K) \to C_*(K))$$

we find that for generic points in V_K

$$\mathrm{rank}(CH_0^{\mathrm{lin}}) = 0, \quad \mathrm{rank}(CH_1^{\mathrm{lin}}) = 1, \quad \mathrm{rank}(CH_2^{\mathrm{lin}}) = 1$$

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- Furthermore, if b generates CH_1^{lin} then the count of disks passing through the reference curve ξ is generically non-zero.

Topological recursion

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- We now describe the topological recursion. At infinity there are only disks. All higher genus curves are "formal" and can be computed via these disks and linking numbers.
- We consider curves of index 0 and 1. A curve has type (n, χ) if it has n positive degree 0 punctures and Euler characteristic χ. An index 0 curve attached to an index 1 curve has attached type (n_0, n_1, χ) if it is attached via n_0 positive punctures and chain insertions and has n_1 free positive degree 0 punctures and Euler characteristic χ.

Topological recursion

- Assume inductively we know the counts of index 0 curves of type (n, χ) for $-\chi + n < r$. Pick a generator b of CH_1^{lin} and consider the boundary of index 1 curves of type $(0, r)$ with positive puncture at b.

Topological recursion

- Assume inductively we know the counts of index 0 curves of type (n, χ) for $-\chi + n < r$. Pick a generator b of CH_1^{lin} and consider the boundary of index 1 curves of type $(0, r)$ with positive puncture at b.

- The broken curves in the boundary with attached curve of type $(1, 0, r)$ are all attached at an insertion (the ones attached at a do not contribute since b is a cycle in Ch^{lin}). The contribution is

$$B(e^x, Q) \cdot F_0^r, \quad B \neq 0.$$

  By the inductive assumption we can then solve for F_0^r in terms of earlier curves and curves at infinity.

Topological recursion

- For curves of type $(j, r-j)$, $j > 0$ take a positive puncture at a_j and pick a primitive b_j of a_j in the linearized complex, study the boundary of index 1 curves of type $(j-1, r-j)$ with positive puncture at b_j to see that we can express it in terms of less complex curves.